CCGPS Frameworks Student Edition

Mathematics

CCGPS Analytic Geometry
Unit 7: Applications of Probability

Dr. John D. Barge, State School Superintendent
"Making Education Work for All Georgians"

Unit 7
Applications of Probability

Table of Contents

OVERVIEW 3
STANDARDS ADDRESSED IN THIS UNIT 3
KEY STANDARDS 3
RELATED STANDARDS 4
ENDURING UNDERSTANDINGS 8
CONCEPTS/SKILLS TO MAINTAIN 9
SELECTED TERMS AND SYMBOLS 9
TASKS
How Odd? 11
The Conditions are Right 21
The Land of Independence 31

OVERVIEW

In this unit, students will:

- take their previously acquired knowledge of probability for simple and compound events and expand that to include conditional probabilities (events that depend upon and interact with other events) and independence.
- be exposed to elementary set theory and notation (sets, subsets, intersection and unions).
- use their knowledge of conditional probability and independence to make determinations on whether or not certain variables are independent.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under “Evidence of Learning” be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

KEY STANDARDS

Understand independence and conditional probability and use them to interpret data

MCC9-12.S.CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (“or,” “and,” “not”). ★

MCC9-12.S.CP.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★

MCC9-12.S.CP.3 Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. ★

MCC9-12.S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. ★

MCC9-12.S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. ★

Use the rules of probability to compute probabilities of compound events in a uniform probability model

MCC9-12.S.CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. ★

MCC9-12.S.CP.7 Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model. ★

RELATED STANDARDS

Investigate chance processes and develop, use, and evaluate probability models.

MCC7.SP.5 Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.

MCC7.SP.8a Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.

MCC7.SP.8b Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., "rolling double sixes"), identify the outcomes in the sample space which compose the event.

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive

reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2 Reason abstractly and quantitatively. Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

3 Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and

respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. Common Core State Standards for Mathematics

5 Use appropriate tools strategically. Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a

website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 Attend to precision. Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8 Look for and express regularity in repeated reasoning. Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics should engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word “understand” are often especially good opportunities to connect the practices to the content. Students who do not have an understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a missing mathematical knowledge effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential “points of intersection” between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

ENDURING UNDERSTANDINGS

- Use set notation as a way to algebraically represent complex networks of events or real world objects.
- Represent everyday occurrences mathematically through the use of unions, intersections, complements and their sets and subsets.
- Use Venn Diagrams to represent the interactions between different sets, events or probabilities.
- Find conditional probabilities by using a formula or a two-way frequency table.
- Understand independence as conditional probabilities where the conditions are irrelevant.
- Analyze games of chance, business decisions, public health issues and a variety of other parts of everyday life can be with probability.
- Model situations involving conditional probability with two-way frequency tables and/or Venn Diagrams.
- Confirm independence of variables by comparing the product of their probabilities with the probability of their intersection.

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

- Understand the basic nature of probability
- Determine probabilities of simple and compound events
- Organize and model simple situations involving probability
- Read and understand frequency tables

SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website.

- **Complement:** Given a set A, the complement of A, denoted $\overline{A}$ or A', is the set of elements that are not members of A.

- **Conditional Probability:** The probability of an event A, given that another event, B, has already occurred; denoted $P(A|B)$.

- **Dependent Events:** Two or more events in which the outcome of one event affects the outcome of the other event or events.

- **Element:** A member or item in a set.

- **Independent Events:** Events whose outcomes do not influence each other.

- **Intersection of Sets:** The set of all elements contained in all of the given sets, denoted $\cap$.

- **Outcome:** A possible result of an experiment.

- **Sample Space:** The set of all possible outcomes from an experiment.

- **Set:** A collection of numbers, geometric figures, letters, or other objects that have some characteristic in common.

- **Subset:** a set in which every element is also contained in a larger set.

- **Union of Sets:** The set of all elements that belong to at least one of the given two or more sets denoted $\cup$.

- **Venn Diagram:** A picture that illustrates the relationship between two or more sets.

How Odd?

MCC9-12.S.CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not"). ★

MCC9-12.S.CP.7 Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model. ★

In middle school mathematics, you took a first look at probability models. You most likely solved problems that involved selecting cards, spinning a spinner, or rolling die to find the likelihood that an event occurs. In this task you will build upon what you already know. You will start with an introduction to **set theory** (a way to algebraically represent different mathematical objects). This will allow you later on in this unit to better explore two branches of probability theory: conditional probability and independence. Through these topics you will be able to uncover how data analysis and probability can help inform us about many aspects of everyday life.

Part 1 – For this task you will need a pair of six-sided dice. In Part 1, you will be concerned with the probability that one (or both) of the dice show odd values.

1. Roll your pair of dice 30 times, each time recording a success if one (or both) of the dice show an odd number and a failure if the dice do not show an odd number.

Number of Successes	Number of Failures

2. Based on your trials, what would you estimate the probability of two dice showing at least one odd number? Explain your reasoning.

3. You have just calculated an *experimental probability*. 30 trials is generally sufficient to estimate the *theoretical probability*, the probability that you expect to happen based upon fair chance. For instance, if you flip a coin ten times you expect the coin to land heads and tails five times apiece; in reality, we know this does not happen every time you flip a

coin ten times.

a. A lattice diagram is useful in finding the theoretical probabilities for two dice thrown together. An incomplete lattice diagram is shown to the right. Each possible way the two dice can land, also known as an **outcome**, is represented as an ordered pair. (1, 1) represents each die landing on a 1, while (4, 5) would represent the first die landing on 4, the second on 5. Why does it have 36 spaces to be filled?

Dice Lattice

(1,1)	(1,2)	(1,3)	(,)	(,)	(,)
(2,1)	(,)	(,)	(,)	(,)	(,)
(,)	(,)	(,)	(,)	(,)	(,)
(,)	(,)	(,)	(,)	(,)	(,)
(,)	(,)	(,)	(,)	(,)	(,)
(,)	(,)	(,)	(,)	(,)	(,)

b. Complete the lattice diagram for rolling two dice.

The 36 entries in your dice lattice represent the *sample space* for two dice thrown. The sample space for any probability model is all the possible outcomes.

c. It is often necessary to list the sample space and/or the outcomes of a set using *set notation*. For the dice lattice above, the set of all outcomes where the first roll was a 1 can be listed as: {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)}. This set of outcomes is a **subset** of the set because all of the *elements* of the subset are also contained in the original set. Give the subset that contains all elements that sum to 9.

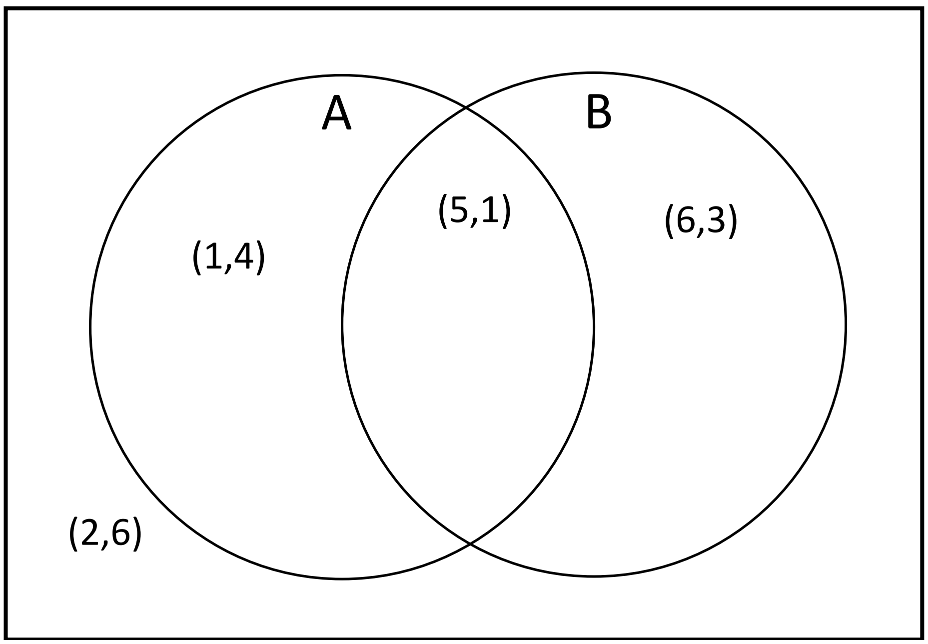
d. What is the probability that the sum of two die rolled will be 9?

e. Using your lattice, determine the probability of having at least one of the two dice show an odd number.

4. The different outcomes that determine the probability of rolling odd can be visualized using a Venn Diagram, the beginning of which is seen below. Each circle represents the possible ways that each die can land on an odd number. Circle A is for the first die landing on an odd number and circle B for the second die landing on odd. The circles

overlap because some rolls of the two dice are successes for both dice. In each circle, the overlap, and the area outside the circles, one of the ordered pairs from the lattice has been placed. $(1,4)$ appears in circle A because the first die is odd, $(6,3)$ appears in circle B because the second die is odd, $(5,1)$ appears in both circles at the same time (the overlap) because each die is odd, and $(2,6)$ appears outside of the circles because neither dice is odd.

a. Finish the Venn Diagram by placing the remaining 32 ordered pairs from the dice lattice in the appropriate place.



b. How many outcomes appear in circle A? (Remember, if ordered pairs appear in the overlap, they are still within circle A).

c. How many outcomes appear in circle B?

d. The portion of the circles that overlap is called the **intersection**. The notation used for intersections is $\cap$. For this Venn Diagram the intersection of A and B is written $A \cap B$ and is read as "A intersect B" or "A and B." How many outcomes are in $A \cap B$?

e. When you look at different parts of a Venn Diagram together, you are considering the **union** of the two outcomes. The notation for unions is $\cup$, and for this diagram the union of A and B is written $A \cup B$ and is read "A union B" or "A or B." In the Venn Diagram you created, $A \cup B$ represents all the possible outcomes where an odd number shows. How many outcomes are in the union?

f. Record your answers to b, c, d, and e in the table below.

b. Circle A	c. Circle B	d. $A \cap B$	e. $A \cup B$

g. How is your answer to e related to your answers to b, c, and d?

h. Based on what you have seen, make a conjecture about the relationship of A, B, $A \cup B$ and $A \cap B$ using notation you just learned.

i. What outcomes fall outside of $A \cup B$ (outcomes we have not yet used)? Why haven't we used these outcomes yet?

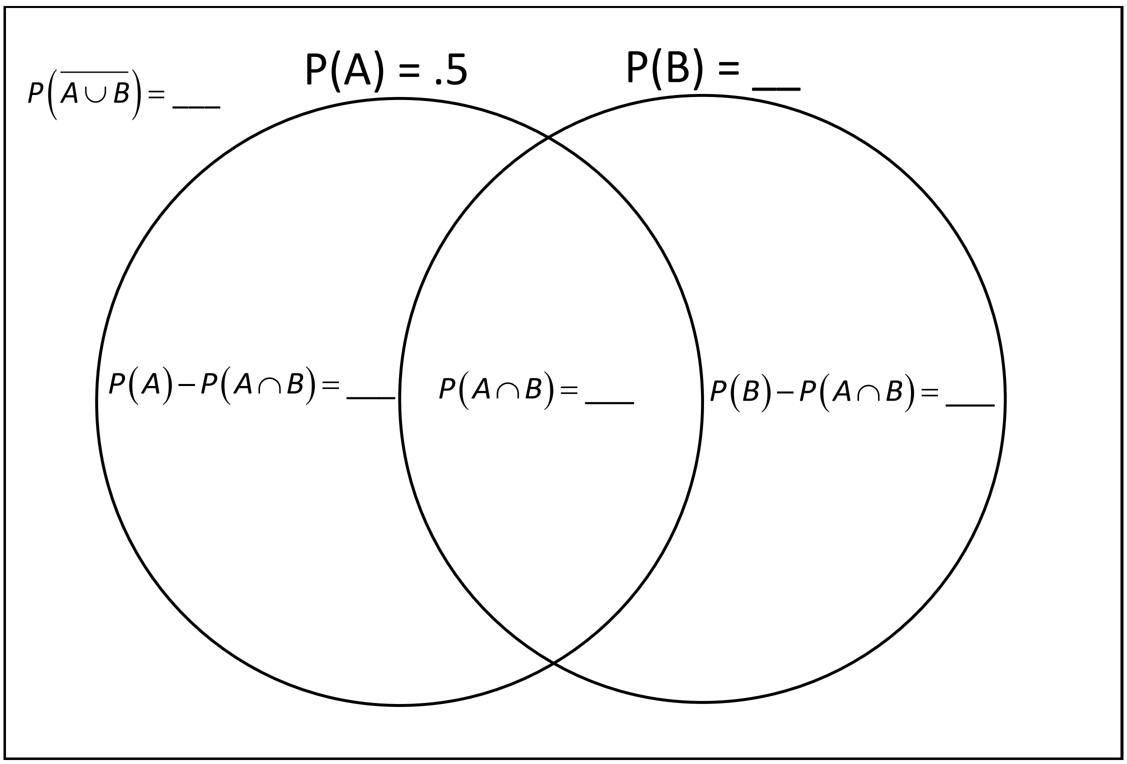
In a Venn Diagram the set of outcomes that are *not* included in some set is called the complement of that set. The notation used for the complement of set A is $\overline{\text{A}}$, read "A bar", or ~A, read "not A". For example, in the Venn Diagram you completed above, the outcomes that are outside of $A \cup B$ are denoted $\overline{\text{A} \cup \text{B}}$.

j. Which outcomes appear in $\overline{\text{A}} - \text{B}$?

k. Which outcomes appear in $\overline{B} - \left(\overline{A \cup B}\right)$?

5. The investigation of the Venn Diagram in question 4 should reveal a new way to see that the probability of rolling at least one odd number on two dice is $\frac{27}{36}=\frac{3}{4}$. How does the Venn diagram show this probability?

6. Venn Diagrams can also be drawn using probabilities rather than outcomes. The Venn Diagram below represents the probabilities associated with throwing two dice together. In other words, we will now look at the same situation as we did before, but with a focus on probabilities instead of outcomes.



a. Fill in the remaining probabilities in the Venn Diagram.

b. Find $P(A \cup B)$ and explain how you can now use the probabilities in the Venn Diagram rather than counting outcomes.

c. Use the probabilities in the Venn Diagram to find $P\left(\overline{B}\right)$.

d. What relationship do you notice between $P(B)$ and $P\left(\overline{B}\right)$? Will this be true for any set and its complement?

Part 2 – Venn Diagrams can also be used to organize different types of data, not just common data sets like that generated from rolling two dice. In this part of the task, you'll have an opportunity to collect data on your classmates and use a Venn Diagram to organize it.

1. Music is a popular topic amongst high school students, but one in which not all can agree upon. Let's say we want to investigate the popularity of different genres of music in your math class, particularly, Hip Hop and Country music. What genre of music do you enjoy listening to: Hip Hop, Country, or Neither?

2. Each student should identify themselves by their 3 initials (first, middle, last). Any student who listens to both Country and Hip Hop may be listed in both categories. Record results of the class poll in the table.

Hip Hop (HH)	Country (C)	Neither (N)

3. Draw a Venn Diagram to organize your outcomes. (*Hint: Students listed in both the Hip Hop and Country categories should be identified first prior to filling in the diagram.)*

4. Find $\mathrm{P(HH)}$.

5. Find $\mathrm{P}\left(\overline{\mathrm{C}}\right)$.

6. Find $\mathrm{P(HH \cap C)}$.

7. Find $\mathrm{P(HH \cup C)}$.

8. In part 1, you found the relationship between A, B, $\mathrm{A \cup B}$, and $\mathrm{A \cap B}$ to be $\mathrm{A \cup B = A + B - A \cap B}$. In a similar way, write a formula for $\mathrm{P(A \cup B)}$.

9. Now find $\mathrm{P(HH \cup C)}$ using the formula instead of the Venn Diagram. Did you get the same answer as you did in f above?

10. In what situation might you be forced to use the formula instead of a Venn Diagram to calculate the union of two sets?

Part 3 – Now that you have had experience creating Venn Diagrams on your own and finding probabilities of events using your diagram, you are now ready for more complex Venn Diagrams.

1. In this part of the task, you will be examining data on the preference of social networking sites based on gender. Again, you will collect data on students in your class, record the data in a *two-way frequency table*, and then create a Venn Diagram to organize the results of the poll. Which social networking site do you prefer?

2. Record results from the class poll in the table.

	Twitter (T)	Facebook (FB)
Female (F)		
Male (M)		

3. Draw a Venn Diagram to organize your outcomes. *(Hint: Notice that male and female will not overlap and neither will Twitter and Facebook).*

4. Find $P(T \cup M)$.

5. What is another way to write the probability of $P(T \cup M)$ using a complement?

6. Find $P(\overline{FB} \cap F)$.

7. Find $P(T \cap M) + P(\overline{T \cup M})$.

The Conditions are Right

MCC9-12.S.CP.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent. ★

MCC9-12.S.CP.3 Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B. ★

MCC9-12.S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. ★

MCC9-12.S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. ★

MCC9-12.S.CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model. ★

Imagine the last time you entered to win a raffle at a fair or carnival. You look at your ticket, 562104. As they begin to call off the winning ticket, you hear 562, but everyone has the same first 3 digits. Then 1 and 0 are called off. You know that excited feeling you get? Did you know there is a lot of math behind that instinct you feel that you might just win the prize? Now imagine those times when you are waiting to get your latest grade back on your English test. You're really not sure how you did, but as your teacher starts to talk about test results, her body language just isn't positive. She keeps saying things like "well, you guys tried hard." Again, there is significant math happening behind that sinking feeling you now have. In this task, you will be investigating how probability can be used to formalize the way real-life conditions change the way we look at the world.

Part 1 - A Game of Pig

To begin this task, you and your team members will compete in a dice game called Pig. The object of the game is to score the most points after 10 rounds of dice rolls. Your score is equal to the sum of all the dice that you roll. If you roll 5 then 5 then 3 then 2 your score is at 15. Your turn starts with a single die roll. You are allowed to keep rolling with the following restrictions:

- If you roll 6 at any time, another die is added to your pool. After the first 6, you will have two die to roll, after the second 6, you will have three to roll. Keep in mind if you roll more than one 6, more than one die is added.

- If you roll 1 at any time, your turn is immediately over, and your score for that turn is 0. It does not matter if it is the first roll or the twentieth.
- You may stop your turn after any single roll, record your score, and pass play to the next player.

You can keep score below. Play a few games, and while you play try to take note of successful strategies.

Players / Round				
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
Total				

1. Regardless of who won, what kind of strategies were most successful? Least successful? Explain why you think so.

2. How does your strategy change as you roll more 6's? How many dice is too dangerous to keep rolling?

3. How would your strategy change if you only lost if you rolled at least two 1's at the same time?

Part 2 – An Introduction to Conditional Probability

As you were able to see by playing Pig, the fact that the probability in a given situation can change greatly affects how a situation is approached and interpreted. This sort of idea is prevalent across society, not just in games of chance. Knowledge of **conditional probability** can inform us about how one event or factor affects another. Say-No-To-Smoking campaigns are vigilant in educating the public about the adverse health effects of smoking cigarettes. This motivation to educate the public has its beginnings in data analysis. Below is a table that represents a sampling of 500 people. Distinctions are made on whether or not a person is a smoker and whether or not they have ever developed lung cancer. Each number in the table represents the number of people that satisfy the conditions named in its row and column.

	Has been a smoker for 10+ years	Has not been a smoker
Has not developed lung cancer	202	270
Has developed lung cancer	23	5

1. How does the table indicate that there is a connection between smoking and lung cancer?

2. Using the 500 data points from the table, you can make reasonable estimates about the population at large by using probability. 500 data values is considered, statistically, to be large enough to draw conclusions about a much larger population. In order to investigate the table using probability, use the following outcomes:

S – The event that a person is a smoker
L – The event that a person develops lung cancer

Find each of these probabilities (write as percentages):

a) $P\left(S\right)$

b) $P\left(\overline{S}\right)$

c) $P\left(L\right)$

d) $P\left(\overline{L}\right)$

e) $P(L \cap S)$

f) $P(\overline{S} \cap \overline{L})$

g) $P(\overline{S} \cap L)$

h) $P(S \cap \overline{L})$

i) $P(S \cup L)$

j) $P(\overline{S} \cup \overline{L})$

3. In order to use probability to reinforce the connection between smoking and lung cancer, you will use calculations of *conditional probability*.

a) By considering only those people who have been smokers, what is the probability of developing lung cancer?

b) Compare the value in 3a to the one for *P(L)* in 2c. What does this indicate?

c) You should be able to confirm that a non-smoker is less likely to develop lung cancer. By considering only non-smokers, what is the probability of developing lung cancer?

4. When calculating conditional probability, it is common to use the term "given." In question 3a, you have calculated the probability of a person developing lung cancer given that they are a smoker. The condition (or, "given") is denoted with a single, vertical bar separating the probability needed from the condition. The probability of a person developing lung cancer given that they are a smoker is written $P(L \mid S)$.

a) Rewrite the question from 3c using the word "given."

b) Write the question from 3c using set notation.

5. Find the probability that a person was a smoker given that they have developed lung cancer and represent it with proper notation.

6. Find the probability that a given cancer-free person was not a smoker and represent it with proper notation.

7. How does the probability in number 6 compare to $P\left(\overline{L} \mid \overline{S}\right)$? Are they the same or different and how so?

8. Based upon finding the conditional probabilities make an argument that supports the connection between smoking and lung cancer.

Part 3 – A Formula for Conditional Probability

The formulaic definition of conditional probability can be seen by looking at the different probabilities you calculated in part 2. The formal definition for the probability of event *A* given event *B* is the chance of both events occurring together with respect to the chance that *B* occurs. As a formula,

Probability of *A* given *B*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

In part 2 you found that $P(S) = \frac{225}{500}$ and $P(L \cap S) = \frac{23}{500}$. Using the formula for conditional probability is another way to determine that $P(L|S) = \frac{23}{225}$:

$$P(L|S) = \frac{P(L \cap S)}{P(S)} = \frac{23/500}{225/500} = \frac{23/\cancel{500}}{225/\cancel{500}} = \frac{23}{225}$$

1. Using the same approach that is shown above, show that the conditional probability formula works for $P(\overline{S}|\overline{L})$.

2. For two events *S* and *Q* it is known that $P(Q) = .45$ and $P(S \cap Q) = .32$. Find $P(S|Q)$.

3. For two events *X* and *Y* it is known that $P(X) = \frac{1}{5}$ and $P(X \cap Y) = \frac{2}{15}$. Find $P(Y|X)$.

4. For two events *B* and *C* it is known that $P(C|B) = .61$ and $P(C \cap B) = .48$. Find $P(B)$.

5. For two events *V* and *W* it is known that $P(W) = \frac{2}{9}$ and $P(V|W) = \frac{5}{11}$. Find $P(V \cap W)$.

6. For two events *G* and *H* it is known that $P(H|G) = \frac{5}{14}$ and $P(H \cap G) = \frac{1}{3}$. Explain why you cannot determine the value of $P(H)$.

Part 4 – Box Office

Movie executives collect lots of different data on the movies they show in order to determine who is going to see the different types of movies they produce. This will help them make decisions on a variety of factors from where to advertise a movie to what actors to cast. Below is a two-way frequency table that compares the preference of *Harry Potter and the Deathly Hallows* to *Captain America: The First Avenger* based upon the age of the moviegoer. 200 people were polled for the survey.

	Prefers *Harry Potter*	Prefers *Captain America*
Under the age of 30	73	52
Age 30 or above	20	55

Define each event in the table using the following variables:

H – A person who prefers *Harry Potter and the Deathly Hallows*
C – A person who prefers *Captain America: The First Avenger*
Y – A person under the age of 30
E – A person whose age is 30 or above

1. By looking at the table, but without making any calculations, would you say that there is a relationship between age and movie preference? Why or why not?

2. Find the following probabilities. In terms of movie preference, explain what each probability—or probabilities together in the case of b, c, and d—would mean to a movie executive.
   a. $P(E)$
   b. $P(H)$ and $P(C)$
   c. $P(C \mid Y)$ and $P(H \mid Y)$
   d. $P(E \mid C)$ and $P(Y \mid C)$

3. Summarize what a movie executive can conclude about age preference for these two movies through knowing the probabilities that you have found.

Part 5 – ICE CREAM

The retail and service industries are another aspect of modern society where probability's relevance can be seen. By studying data on their own service and their clientele, businesses can make informed decisions about how best to move forward in changing economies. Below is a table of data collected over a weekend at a local ice cream shop, Frankie's Frozen Favorites. The table compares a customer's flavor choice to their cone choice.

Frankie's Frozen Favorites	Chocolate	Butter Pecan	Fudge Ripple	Cotton Candy
Sugar Cone	36	19	34	51
Waffle Cone	35	56	35	24

1. By looking at the table, but without making any calculations, would you say that there is a relationship between flavor and cone choice? Why or why not?

2. Find the following probabilities (write as percentages):

a. $P(W)$

b. $P(S)$

c. $P(C)$

d. $P(BP)$

e. $P(FR)$

f. $P(CC)$

3. In order to better investigate the correlation between flavor and cone choice, calculate the conditional probabilities for each cone given each flavor choice. A table has been provided to help organize your calculations.

Frankie's Frozen Favorites	Chocolate	Butter Pecan	Fudge Ripple	Cotton Candy
Sugar Cone	$P(S\mid C)$	$P(S\mid BP)$	$P(S\mid FR)$	$P(S\mid CC)$
Waffle Cone	$P(W\mid C)$	$P(W\mid BP)$	$P(W\mid FR)$	$P(W\mid CC)$

4. Compare and contrast the probabilities you found in question 2 with the conditional probabilities you found in question 3. Which flavors actually affect cone choice? Which do not? How did you make this determination?

5. The relationship that you have observed between chocolate and cone choice (and fudge ripple and cone choice) is called **independence**. Multiple events in probability are said to be independent if the outcome of any one event does not affect the outcome of the others. The fact that $P(S\mid C)$ and $P(W\mid C)$ are approximately equal to each other indicates that the choice of cone is in no way affected by the choice of chocolate ice cream. The same is true for fudge ripple. When probabilities change depending on the situation, such as knowing sugar cones are more likely with cotton candy ice cream, the events have a **dependent** relationship. Answer the questions below to ensure you understand this new terminology:

a. Explain whether or not flipping a coin twice would be considered a set of two independent events.

b. A game is played where marbles are pulled from a bag, 8 of which are red and 2 are white. You score by pulling marbles from the bag, one at a time, until you pull a white marble. Are the events in this game independent or dependent? Why?

c. Explain whether or not the dice game of Pig you played represents independent or dependent events.

6. Consider the statement, “the probability that a sugar cone is chosen given that chocolate ice cream is chosen.” The desired probability relates to a sugar cone, but this choice is *independent* of the choice of chocolate. That is to say the statement “the probability that a sugar cone is chosen” is no different when “given that chocolate ice cream is chosen” is removed. Thus, we can say $P(S \mid C) = P(S)$. Which other parts of the table from question 3 can be written in a simpler way?

7. For independent events, the conditional probability formula, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, becomes $P(A) = \dfrac{P(A \cap B)}{P(B)}$. Solve this equation for $P(A \cap B)$ and place it in the box below.

Probability of Independent Events *A* and *B*
$P(A \cap B) =$

Notice that for *independent events*, the probability of two events occurring together is simply the product of each event’s individual probability.

8. To conclude, let’s go back and revisit Frankie’s Frozen Favorites. In this problem, you discovered pairs of events that were independent of one another by comparing their conditional probabilities to the probabilities of single events. Because conditions do not need to be considered when calculating probabilities of two independent events, we arrived at the formula above.

a. Use the formula to verify that Fudge Ripple and Waffle Cone are independent events.

b. Explain in full why this formula will *not* accurately calculate $P(CC \cap W)$.

The Land of Independence

MCC9-12.S.CP.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

MCC9-12.S.CP.3 Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.

MCC9-12.S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.

MCC9-12.S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.

Part 1 – Confirming Independence

By developing a full picture of conditional probability in the previous task, you were able to conclude that events that occur without regard to conditions, independent events, are defined by the equation $P(A \cap B) = P(A) \cdot P(B)$. This equation is known as *necessary and sufficient*. It works exactly like a biconditional statement: two events *A* and *B* are independent if and only if the equation $P(A \cap B) = P(A) \cdot P(B)$ is true.

1. Based upon the definition of independence, determine if each set of events below are independent.

   a. $P(A) = 0.45,\ P(B) = 0.30,\ P(A \cap B) = 0.75$

   b. $P(A) = 0.12,\ P(B) = 0.56,\ P(A \cap B) = 0.0672$

   c. $P(A) = \frac{4}{5},\ P(B) = \frac{3}{8},\ P(A \cap B) = \frac{7}{40}$

   d. $P(A) = \frac{7}{9},\ P(B) = \frac{3}{4},\ P(A \cap B) = \frac{7}{12}$

2. Determine the missing values so that the events *A* and *B* will be independent.

   a. $P(A) = 0.55,\ P(B) =$ ______ $,\ P(A \cap B) = 0.1375$

b. $P(A) = \_\_\_\_\_\_$, $P(B) = \frac{3}{10}, P(A \cap B) = \frac{1}{7}$

Part 2 – Independence and Inference

With knowledge of probability and statistics, statisticians are able to make *statistical inferences* about large sets of data. Based upon what you have learned in this unit, you have the knowledge necessary to make basic inferences.

Much of the data collected every 10 years for the Census is available to the public. This data includes a variety of information about the American population at large such as age, income, family background, education history and place of birth. Below you will find three different samples of the Census that looks at comparing different aspects of American life. Your job will be to use your knowledge of conditional probability and independence to make conclusions about the American populace.

Gender vs. Income – Has the gender gap closed in the world today? Are men and women able to earn the same amount of money? The table below organizes income levels (per year) and gender.

	Under $10,00	Between $10,000 and $40,000	Between $40,000 and $100,000	Over $100,000	
Male	15	64	37	61	
Female	31	73	14	58	

By finding different probabilities from the table above, make a determination about whether or not income level is affected by gender. Investigate whether your conclusion is true for all income levels. Show all the calculations you use and write a conclusion using those calculations.

Bills vs. Education – When you grow up, do you think the amount of schooling you have had will be at all related to the amount of money you have to pay out in bills each month? Below is a table that compares two variables: the highest level of education completed (below a high school diploma, a high school diploma, or a college degree) and the amount paid for a mortgage or rent each month.

	Pays under $500	Pays between $500 and $1000	Pays over $1000	
Below high school	57	70	30	
High school diploma	35	47	11	
College degree	24	62	40	

By determining the probabilities of each education level and the probabilities of housing costs, you should be able to decide whether or not these two variables are independent. Show all the calculations you use, and write a conclusion about the *interdependence* of these two variables.

Gender vs. Commute – What else might gender affect? Is your commute to work related to whether or not you are male or female? The data below allows you to investigate these questions by presenting gender data against the minutes needed to commute to work each day.

	Under 30 minutes	Between 30 minutes and an hour	Over an hour	
Male	65	24	15	
Female	64	22	7	

By finding various probabilities from the table above, decide whether or not a person's gender is related to their commute time to work. Write your conclusion below and include any relevant calculations.